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MECHANICS

The Hula-Hoop Problem

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A hula-hoop is sports equipment, which became popular in the 1960s, and is a thin-walled hoop that goes around the athlete's waist. For spinning a hula-hoop, the athlete's waist makes periodic motions in the horizontal plane resulting in stable rotations. In [1] the periodic motion of the athlete's waist along one axis was considered, and the hula-hoop problem was reduced to the problem of a pendulum with a vibrating suspension point in the absence of gravity. The stable mode of pendulum rotation with an average angular velocity equal to the excitation frequency was found approximately, and the conditions of stability of this mode were obtained. In [2] the same mode of rotation was found for this pendulum by the method of averaging in the second approximation, and its stability conditions were investigated. Stable hula-hoop rotation for the periodic excitation along two axes was studied by the method of direct separation of motion in [3].

In this study, we considered the hula-hoop excitation along two axes corresponding to an elliptic trajectory of the motion of the athlete's waist. For identical excitation amplitudes, exact solutions corresponding to the hula-hoop rotation with a constant angular velocity equal to the excitation frequency are obtained. The stability of these solutions is investigated. The conditions of the inseparable hula-hoop rotation, both stable and unstable, are derived.

The case of close excitation amplitudes corresponding to the motion of the athlete's waist along an ellipse close to a circle is considered. The solutions of the problem on stable hula-hoop rotation in the first, second, and third approximation are obtained by the averaging method. The comparison with the numerical solution obtained with high accuracy shows that the third approximation practically coincides with it. The conditions of coexistence of stable rotation modes with opposite directions are obtained. An interesting case when the athlete's waist rotates oppositely to the rotation of the hula-hoop is investigated.

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1. BASIC RELATIONS

We assume that the athlete's waist represents a circle with the center at the point O', the motion of which in time t is described by the ellipse $x = a\sin\omega t$, $y = b\cos\omega t$ with the center at the origin of coordinates O, semiaxes a, b, and the excitation frequency ω (Fig. 1). Without restriction of generality, we consider that $a \geq |b|$ and $\omega > 0$. For these assumptions, the sign of b determines the direction of motion of the center of the waist along an ellipse. The center of the waist moves clockwise if $b > 0$ and counter-clockwise if $b < 0$, and the trajectory degenerates into a straight-line if $b = 0$.

The equations of the hula-hoop motion in the system of coordinates connected to the athlete's waist taking into account viscous friction have the form

$$I_c\ddot{\theta} + k\dot{\theta} = -F_{\mathrm{T}}R, \tag{1}$$

$$m(R-r)\ddot{\varphi} = m(\ddot{x}\sin\varphi + \ddot{y}\cos\varphi) + F_{\mathrm{T}}, \tag{2}$$

$$m(R-r)\dot{\varphi}^2 = N + m(\ddot{x}\cos\varphi - \ddot{y}\sin\varphi), \tag{3}$$

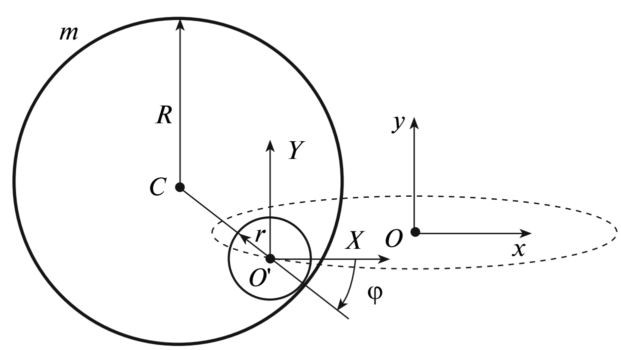

Fig. 1. Motion of hula-hup.

where θ is the hula-hoop rotation angle with respect to the center of mass C, φ is the angle between the axis x and the radius CO', $I_c = mR^2$ is the moment of inertia of the hula-hoop with respect to the center of mass, F_T is the force of friction, m is the hula-hoop mass, R is its radius, k is the viscosity coefficient, r is the radius of the athlete's waist, and N is the normal force. Equation (1) describes the hula-hoop rotation around the center of mass, and Eqs. (2) and (3) are the equations of the hula-hoop motion in the projections to the longitudinal and transverse directions to the CO'.

Considering that there is no slip at the contact point, we have the following kinematic constraint:

$$(R-r)\dot{\varphi} = R\dot{\theta}. \tag{4}$$

Excluding the force of friction F_T from Eqs. (1) and (2) and using Eq. (4), we obtain the equation with respect to the angle φ as

$$\ddot{\varphi} + \frac{k}{2mR^2}\dot{\varphi} + \frac{\omega^2(a\sin\omega t\sin\varphi + b\cos\omega t\cos\varphi)}{2(R-r)} = 0. \tag{5}$$

From Eq. (3), we express the normal force N and write the condition $N > 0$ as

$$m(R-r)\dot{\varphi}^2 + m\omega^2(a\sin\omega t\cos\varphi - b\cos\omega t\sin\varphi) > 0, \tag{6}$$

which means that the hula-hoop does not separate from the athlete's waist during the motion.

We introduce new time $\tau = \omega t$ and the dimensionless parameters

$$\gamma = \frac{k}{2mR^2\omega}, \quad \mu = \frac{a+b}{4(R-r)}, \quad \varepsilon = \frac{a-b}{4(R-r)}. \tag{7}$$

Then Eqs. (5), (6) after trigonometric calculations take the form

$$\ddot{\varphi} + \gamma\dot{\varphi} + \mu\cos(\varphi-\tau) = \varepsilon\cos(\varphi+\tau), \tag{8}$$

$$\dot{\varphi}^2 - 2\mu\sin(\varphi-\tau) + 2\varepsilon\sin(\varphi+\tau) > 0, \tag{9}$$

where the dot designates the derivative in time τ. These equations include three dimensionless parameters: the damping factor γ, and the dimensionless excitation parameters μ and ε. The relation between the parameters μ and ε determines the size and shape of the ellipse—the trajectory of the motion of the athlete's waist. From Eq. (7), it can be seen that it is the straight-line for $\mu = \varepsilon$ and the circle for $\varepsilon = 0$ or $\mu = 0$, the center of the waist moving clockwise for $\mu > \varepsilon$ and counter-clockwise for $\mu < \varepsilon$. From the assumption $a \geq |b|$ and Eqs. (7), it follows that the parameters μ and ε are nonnegative.

2. EXACT SOLUTION OF THE UNPERTURBED EQUATION

We find the mode of hula-hoop rotation for the circular excitation $\varepsilon = 0$ ($a = b$) for an arbitrary damping factor γ. In this case, we call Eq. (8) the unperturbed equation

$$\ddot{\varphi} + \gamma\dot{\varphi} + \mu\cos(\varphi-\tau) = 0, \tag{10}$$

which has the exact rotation solution

$$\varphi = \tau + \psi. \tag{11}$$

The initial rotation phase is determined by the relation

$$\cos\psi = -\frac{\gamma}{\mu}, \tag{12}$$

where we assume that $\mu > 0$. Equality (12) imposes the restriction on the parameters of amplitude and damping in the form

$$|\gamma| \leq \mu. \tag{13}$$

When fulfilling inequality (13), we find

$$\psi = \pm\arccos\left(-\frac{\gamma}{\mu}\right) + 2\pi k, \quad k = 0, 1, \ldots \tag{14}$$

Solutions (11), (14), correspond to hula-hoop rotation with the constant angular velocity equal to the excitation frequency ω.

We investigate the stability of these solutions. For this purpose, we present the angle φ as $\varphi = \tau + \psi + \eta(\tau)$, where $\eta(\tau)$ is a small addition, and substitute it in Eq. (10). Then linearizing in $\eta(\tau)$ and using Eq. (12), we obtain the linear equation

$$\ddot{\eta} + \gamma\dot{\eta} - \mu\sin\psi\eta = 0. \tag{15}$$

According to the Lyapunov stability theorem, solution (11), (14) is asymptotically stable according to the linear approximation if all eigenvalues of linearized Eq. (15) have negative real parts. It follows from the Routh–Hurwitz conditions that they are fulfilled for

$$\gamma > 0, \quad \sin\psi < 0, \tag{16}$$

because it is assumed that $\mu > 0$.

From conditions (16) and Eq. (14), for $\gamma > 0$, it follows that the solution

$$\varphi = \tau + \psi, \quad \psi = -\arccos\left(-\frac{\gamma}{\mu}\right) + 2\pi k, \quad k = 0, 1, \ldots, \tag{17}$$

is asymptotically stable, and the solution

$$\varphi = \tau + \psi, \quad \psi = \arccos\left(-\frac{\gamma}{\mu}\right) + 2\pi k, \quad k = 0, 1, \ldots, \tag{18}$$

is unstable. For negative damping, $\gamma < 0$, both these solutions are unstable.

We check now for exact solutions (11), (14) of rotation condition (9) without separation, which acquires the following form taking into account Eq. (11):

$$1 - 2\mu\sin\psi > 0. \tag{19}$$

Due to Eq. (16) and the assumption $\mu > 0$, inseparability condition (19) is obviously fulfilled for stable solution (17). For unstable solution (18), $\sin\psi > 0$. In this case, we substitute the expression $\sin\psi = \sqrt{1 - \frac{\gamma^2}{\mu^2}}$ following from Eq. (12) into Eq. (19) and obtain the condition of inseparable unstable rotation in the form $\mu^2 < \frac{1}{4} + \gamma^2$. From here taking into account condition (13) and the assumption $\mu > 0$ for solution (18), we obtain the existence condition

$$|\gamma| \le \mu < \sqrt{\frac{1}{4} + \gamma^2}, \tag{20}$$

restricting the excitation amplitude from below and from above.

Thus, solution (17) fulfilling condition

$$0 < \gamma \le \mu \tag{21}$$

corresponds to asymptotically stable inseparable hula-hoop rotation with a constant angular velocity ω, while solution (18) corresponds to unstable inseparable hula-hoop rotation with a constant angular velocity ω if inequality (20) is fulfilled.

We note that the phase ψ of stable solution (17) tends to $-\pi/2$ at $\gamma \to +0$, while the phase of unstable solution (18) tends to $\pi/2$.

3. ASYMPTOTIC SOLUTIONS

For unequal but close amplitudes $a \neq b$ in Eqs. (8), (9), we take ε as a small parameter. Passing to slow variables

$$x_1 = \varphi - \tau, \quad x_2 = \dot{\varphi}, \tag{22}$$

we change Eq. (8) to the form conventional for using the averaging method [2]:

$$\begin{aligned} \dot{x}_1 &= [x_2 - 1], \\ \dot{x}_2 &= -[\gamma x_2 + \mu\cos(x_1)] + \varepsilon\cos(x_1 + 2\tau), \end{aligned} \tag{23}$$

where the expressions in square brackets are small due to Eq. (12) and $x_2 \approx 1$. We consider these expressions as small values of the same order of magnitude as that of ε. In the first, second, and third approximations, we obtain the same averaged equation by the averaging method:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 - 1, \\ \dot{\xi}_2 &= -\gamma\xi_2 - \mu\cos\xi_1, \end{aligned} \tag{24}$$

where ξ_1 and ξ_2 are the averaged variables x_1 and x_2. Excluding ξ_2, we transform Eqs. (24) to the second-order equation

$$\ddot{\xi}_1 + \gamma(1 + \dot{\xi}_1) + \mu\cos\xi_1 = 0. \tag{25}$$

The steady-state solutions of this equation are found from the equality

$$\cos\xi_1 = -\frac{\gamma}{\mu}. \tag{26}$$

From here, we express the averaged variable

$$\xi_1 = \pm\arccos\left(-\frac{\gamma}{\mu}\right) + 2\pi k, \quad k = 0, 1, \ldots. \tag{27}$$

Knowing the averaged variables, we find the solutions for slow variables x_1 and x_2. In the first approximation, we have

$$\begin{aligned} x_1 &= \xi_1, \\ x_2 &= 1 + \frac{\varepsilon}{2}\sin(2\tau + \xi_1). \end{aligned} \tag{28}$$

The solution in the second approximation has the following form:

$$\begin{aligned} x_1 &= \xi_1 - \frac{\varepsilon}{4}\cos(2\tau + \xi_1), \\ x_2 &= 1 + \frac{\varepsilon}{2}\sin(2\tau + \xi_1) + \frac{\varepsilon\gamma}{4}\cos(2\tau + \xi_1). \end{aligned} \tag{29}$$

The solution of Eqs. (23) in the third approximation is

$$\begin{aligned} x_1 &= \xi_1 - \frac{\varepsilon}{4}\cos(2\tau + \xi_1) + \frac{\varepsilon\gamma}{8}\sin(2\tau + \xi_1), \\ x_2 &= 1 + \frac{\varepsilon}{2}\left(1 - \frac{\varepsilon\gamma^2}{4}\right)\sin(2\tau + \xi_1) \\ &\quad + \frac{\varepsilon\mu}{8}\cos(2\tau + 2\xi_1) \\ &\quad + \frac{3\varepsilon\gamma}{8}\cos(2\tau + \xi_1) - \frac{\varepsilon^2}{32}\cos(4\tau + 2\xi_1). \end{aligned} \tag{30}$$

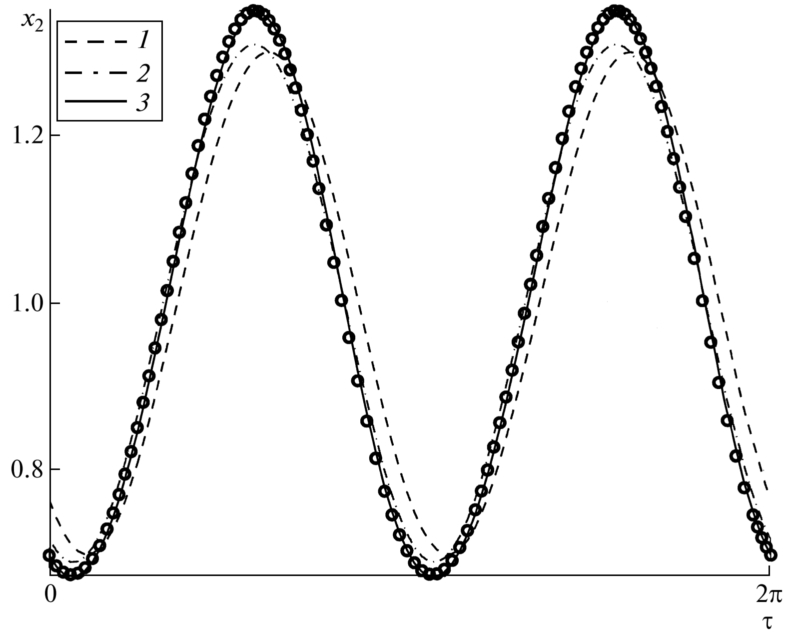

Fig. 2. Comparison of angular velocities x_2 in three successive approximations (28), (29), and (30) with the numerical solution of Eq. (23) for the parameters $\varepsilon = 0.6$, $\mu = 0.8$, and $\gamma = 0.5$. Here and in Fig. 3, *1* is the first approximation, *2* is the second approximation, and *3* is the third approximation. Points represent the numerical solutions.

Passing with the help of Eq. (22) to the variable φ, we obtain the solution of Eq. (8) from Eqs. (28), (29), and (30) in the first approximation as

$$\varphi = \tau + \xi_1, \quad (31)$$

in the second approximation as

$$\varphi = \tau + \xi_1 - \frac{\varepsilon}{4}\cos(2\tau + \xi_1), \quad (32)$$

and in the third approximation as

$$\varphi = \tau + \xi_1 - \frac{\varepsilon}{4}\cos(2\tau + \xi_1) + \frac{\varepsilon\gamma}{8}\sin(2\tau + \xi_1), \quad (33)$$

where ξ_1 is determined from Eq. (27).

The existence and stability conditions of these solutions coincide with the conditions for exact solution (13) and (16) if we assume $\mu > 0$. Thus, solutions (31), (32), and (33), where

$$\xi_1 = -\arccos\left(-\frac{\gamma}{\mu}\right) + 2\pi k, \quad k = 0, 1, \ldots, \quad (34)$$

are asymptotically stable, and solutions (31), (32), and (33), where

$$\xi_1 = \arccos\left(-\frac{\gamma}{\mu}\right) + 2\pi k, \quad k = 0, 1, \ldots, \quad (35)$$

are unstable. From here, it can be seen that solutions (31), (34) and (31), (35) in the first approximation do not differ from solutions (17) and (18) in the zero approximation, while the small oscillations are added to (32), (34), and (35) in the second approximation and solutions (33), (34), and (35) in the third approximation.

We check the inseparability condition for first-approximation solutions (28). Substituting the angle φ and the angular velocity $\dot{\varphi}$ obtained from Eqs. (22) and (28) into inequality (9), we obtain the inseparability condition of rotation in the form

$$1 - 2\mu\sin\xi_1 + 3\varepsilon\sin(2\tau + \xi_1) > 0. \quad (36)$$

This condition is obviously fulfilled for the stable solution (28), (34), at $\varepsilon < 1/3$ due to $\sin\xi_1 < 0$ from Eq. (16) and the assumption $\mu > 0$. For unstable solution (28), (35), it is easy to show that inseparability condition (36) is valid only if the inequality $1 > 2\sqrt{\mu^2 - \gamma^2} + 3\varepsilon$ limiting the excitation amplitude from above is fulfilled. From here taking into account condition (13), the assumptions $\mu > 0$, and $\varepsilon < 1/3$ for solution (28), (35), we obtain the existence condition

$$|\gamma| \le \mu < \sqrt{\left(\frac{1 - 3\varepsilon}{2}\right)^2 + \gamma^2}, \quad (37)$$

restricting the excitation amplitude from below and from above. Condition (37) generalizes condition (20) for the first approximation.

Thus, we obtain the solutions of Eqs. (23) in three approximations (28), (29), and (30) by the averaging method. It should be noted that all solutions are

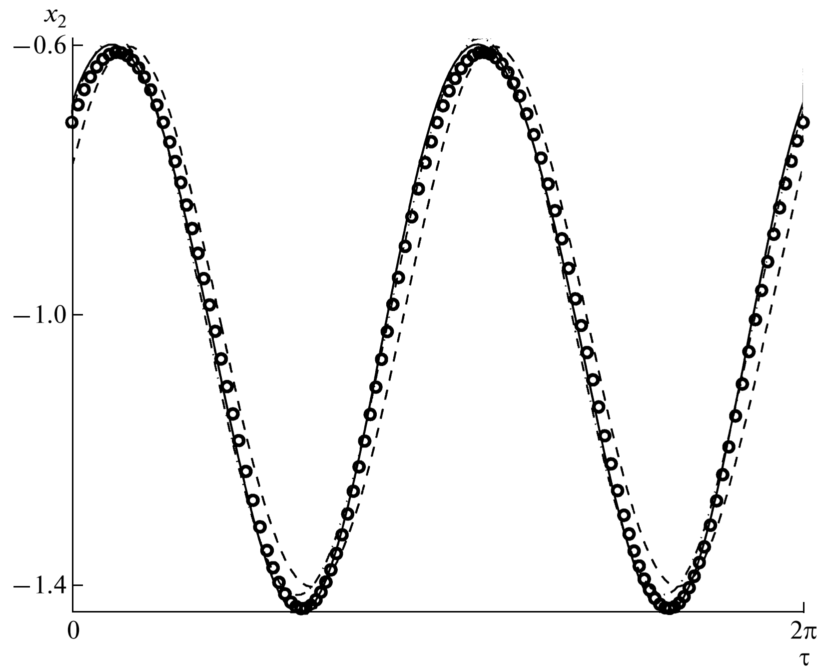

Fig. 3. Reverse rotation. Comparison of the angular velocities x_2 in three successive approximations (41), (42), and (43) with the numerical solution of Eq. (23) at parameters $\varepsilon = 0.6$, $\mu = 0.8$, and $\gamma = 0.5$.

obtained without the assumption of smallness dumping and of excitation amplitudes contrary to those in [1–3]. The graphs of angular velocities x_2 in three different approximations are shown in Fig. 2 for the parameters $\varepsilon = 0.6$, $\mu = 0.8$, and $\gamma = 0.5$. These graphs are compared with the numerical solution of Eqs. (24) with the initial conditions $x_1(0) = -2.2093$ and $x_2(0) = 0.6991$. From Fig. 2, it can be seen that the third approximation is closer to the numerical solution than the first and second approximations.

4. REVERSE ROTATION

It is natural to assume that there are also reverse hula-hoop rotations, i.e., counter-clockwise ones. If we take μ instead of ε as a parameter in Eq. (8), the stable solution of the unperturbed equation proves to be $\varphi = -\tau + \arccos\left(-\frac{\gamma}{\varepsilon}\right) + 2\pi k$, which corresponds to a rotation reverse with respect to that described by Eq. (11). After performing calculations similar to those in Section 2, we obtain the existence conditions of stable reverse rotations for small parameter μ

$$0 < \gamma \le \varepsilon. \tag{38}$$

It should be noted that these conditions can be fulfilled simultaneously with the conditions for forward rotations (21) if the following inequalities are fulfilled:

$$0 < \gamma \le \min\{\mu, \varepsilon\}. \tag{39}$$

Condition (39) is obtained under the assumption that both ε and μ are small. Inseparability condition (9) is always fulfilled for $\varepsilon \ll 1$ and $\mu \ll 1$.

Thus, we obtain coexistence conditions (39) for the stable forward and reverse inseparable rotations when the athlete rotating his waist in one direction along a fixed elliptic trajectory can rotate the hula-hoop in an arbitrary direction depending on the initial conditions.

Substituting expressions from Eqs. (7) into Eq. (39), we obtain the coexistence conditions for the stable forward and reverse rotations in the initial physical parameters:

$$0 < 2k\frac{R-r}{R^2\omega m} \le a - |b|. \tag{40}$$

From here it can be seen that, for fulfilling these conditions, it is necessary that the ellipse along which the center of the athlete's waist moves should not degenerate in the circle.

The coexistence of forward and reverse rotations means that there are stable inseparable solutions, when the athlete's waist rotates oppositely to the hula-hoop rotation. Such a solution is shown in Fig. 3, where the graphs of angular velocities x_2 of reverse rotations in three successive approximations are represented for the same parameters as in Fig. 2. The approximate solutions for reverse rotations are obtained the same way as in Section 3 with the only difference being that the expression for slow variable x_1

in Eq. (22) is replaced by $x_1 = \varphi + \tau$. As a result, we obtain the expressions for the angular velocities in three approximations similar to Eqs. (28), (29), and (30):

$$x_2 = -1 - \frac{\mu}{2}\sin(2\tau - \xi_1), \tag{41}$$

$$x_2 = -1 - \frac{\mu}{2}\sin(2\tau - \xi_1) - \frac{\mu\gamma}{4}\cos(2\tau - \xi_1), \tag{42}$$

$$\begin{aligned} x_2 = &-1 - \frac{\mu}{2}\left(1 - \frac{\mu\gamma^2}{4}\right)\sin(2\tau - \xi_1) \\ &- \frac{\varepsilon\mu}{8}\cos(2\tau - 2\xi_1) - \frac{3\mu\gamma}{8}\cos(2\tau - \xi_1) \\ &+ \frac{\mu^2}{32}\cos(4\tau - 2\xi_1), \end{aligned} \tag{43}$$

where $\xi_1 = \arccos\left(-\frac{\gamma}{\varepsilon}\right) + 2\pi k$ for stable rotations.

These graphs are compared with the numerical solution of Eqs. (23) with the initial conditions $x_1(0) = 2.4984$ and $x_2(0) = -0.7125$. From Fig. 3, it can be seen that the third approximation is very close to the numerical solution in spite of the fact that the small parameter $\mu = 0.8$.

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